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NONLINEAR DEVELOPMENT OF TWO-DIMENSIONAL HYDROELASTIC INSTABILITY IN A TURBULENT BOUNDARY LAYER ON AN ELASTIC COATING

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Nonlinear evolution of hydroelastic instability arising in the flow past a coating of a rubber-type material by a turbulent boundary layer of an incompressible fluid is studied. A nonlinear dispersion equation for two-dimensional, quasi-monochromatic, low-amplitude waves is derived. The Prandtl equations for the mean (over the waviness period) boundary-layer flow are solved in the approximation of local similarity and by direct numerical integration. Evolution of unstable waves in time is studied on the basis of the Landau equation, which is derived separately for the instability of fast waves (flutter) and the quasi-static instability (divergence). The calculation results are compared with available experimental data.

The study of wave generation on elastic coatings in incompressible fluid flows is of interest for using these coatings to decrease the drag and suppress acoustic noise and vibrations [1, 2]. Up to now, the main attention was focused on the linear theory of instability arising upon interaction of elastic coatings of various types with a laminar flow (see the review of literature in [3]). The problem of excitation of finite-amplitude waves was also solved for a laminar flow regime [4, 5].

Generation of waves on elastic coatings in a turbulent boundary layer (TBL) was experimentally studied in some papers [1, 6]. Two basic regimes of generation of hydroelastic waves were found: traveling-wave flutter (TWF) and wave divergence. Reutov and Rybushkina [7] used an algebraic model of vortex viscosity to study linear hydroelastic instability in the TBL, and equations for two-dimensional wave perturbations in the boundary layer were written in curvilinear coordinates. The calculated values of the critical velocity of TWF and wave-divergence origination are in good agreement with the experimental data of [1, 6]. Reutov [8] proposed a numerical model, which allows one to calculate the nonlinear response of the TBL to a wavy flexure of the underlying surface. As in [9], where the interaction of waves on water with an atmospheric TBL was examined, Reutov [8] used a quasi-linear approximation, where the basic nonlinear effects are related to deformation of the profile of the mean (over the waviness period) flow.

In the present paper, which should be considered as a continuation of [7, 8], we study the nonlinear stage of evolution of hydroelastic instability in the TBL on a single-layer coating. The main small parameter of the problem is the slope of the wavy surface $ka \ll 1$ (k and a are the wavenumber and the amplitude of surface flexure). Another limitation of the proposed theory is the fact that the surface flexure has the form of a two-dimensional quasi-monochromatic wave.

In the above-cited experiments [1, 6], the flow velocity could exceed the critical value by several times. Waves with large slopes of the surface were observed. The approach proposed in the present paper allows one to consider the region of small and moderately small supercritical values at which rather weak waves are generated. We note that generation of divergent waves with large slopes of the surface were numerically simulated by Lucey and Carpenter [10]. However, the potential-flow approximation was used in the latter

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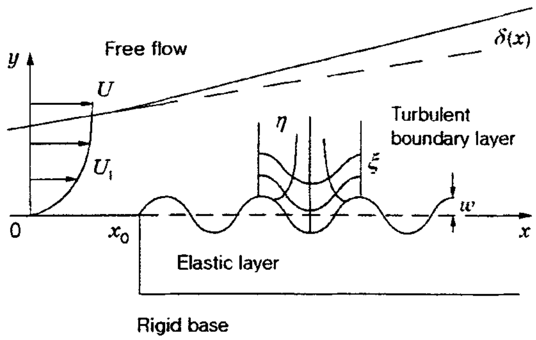

Fig. 1

work, which does not take into account the TBL effect.

1. Nonlinear Dispersion Equation for Small-Amplitude Hydroelastic Waves. Following [8], we describe the TBL flow using a semi-empirical hypothesis of turbulent viscosity and formulate the equations of motion in curvilinear coordinates ξ, η, in which one of the coordinate axes coincides with the profile of the wavy surface. The x and y axes of the Cartesian coordinate system are directed along the flow and normal to the undisturbed surface. The spatial evolution of the TBL above the wavy surface beginning in the cross section $x = x_0$ is schematically shown in Fig. 1 [$\delta(x)$ is the TBL thickness, the dashed curve shows its behavior in the absence of waviness, $U_1(y)$ is the profile of the longitudinal velocity in the TBL over a smooth surface, U is the free-stream velocity, and ξ and η are the coordinate lines of the system of orthogonal curvilinear coordinates over a waviness period]. Separation of the flow into the mean and fluctuating components is performed along the horizontal coordinate lines ($\eta = \text{const}$). The mean flow is assumed to be quasi-parallel, i.e., the scale of expansion of the TBL $L = \delta/(d\delta/dx)$ is much greater than the TBL thickness δ and the waviness scale ($L/\delta \gg 1$ and $kL \gg 1$). Wavy deflections with a small slope ($ka \ll 1$) are considered. An elastic coating made of an incompressible rubber-type material, which is characterized by the density ρ_s and shear modulus G, has a thickness d. The velocity of propagation of transverse waves in the coating material is $c_t = \sqrt{G/\rho_s}$.

The effect of surface shear stresses ignored [1], determination of the TBL response to a wavy flexure of the surface $w(x,t)$ reduces to determination of surface-pressure perturbations $p(x,t)$ generated by this flexure (Fig. 1). The hydrodynamic aspect of the problem is considered in more detail in Sec. 2. At this stage of constructing the dispersion equation for hydroelastic waves, it suffices to take into account the fact that the nonlinear response of the TBL can be found in the quasi-linear approximation. The equations and boundary conditions for the fluctuating (wavy) component of the TBL flow are the same as in the linear problem, but the mean-flow characteristics are found taking into account the influence of wave stresses, which are quadratic in amplitude. The conditions of applicability of the quasi-linear approximation were discussed in [8, 9]. Another important feature of the nonlinear response is that it can be determined without taking into account the TBL expansion history.

The main term of expansion in $ka \ll 1$ for surface displacement and surface-pressure perturbations is represented in the form

$$(w, p) = (1/2)(\hat{w}, \hat{p}) \exp(ik(x - ct)) + \text{c.c.}, \tag{1}$$

where c is the phase velocity and $|\hat{w}| = a$ is the flexure amplitude: the hat denotes the complex amplitude of the wave perturbation (c.c. is the complex conjugate expression). The TBL response to the wavy flexure of the surface is characterized by the complex elasticity of the flow $K_A(c,k) = \hat{p}/\hat{w}$. In the quasi-linear approximation, the elasticity is a function of $(ka)^2$ and may be considered as a virtual elasticity [7]. For small ka, we obtain the dimensionless elasticity of the flow

$$Y = \frac{K_A}{\rho_0 k U^2} \simeq Y_0 + (ka)^2 Y_1, \tag{2}$$

where ρ_0 is the fluid density. The main term of expansion in the right part of (2) Y_0 coincides with the linear elasticity of the flow calculated in [7]; the coefficient Y_1 characterizes the nonlinear properties of the TBL response. According to [8], the condition of applicability of the quasi-linear approximation is the presence of a numerically large coefficient at $(ka)^2$ in Eq. (2): $|Y_1|/|Y_0| \gg 1$, where the characteristic values of the corresponding quantities are taken as $|Y_1|$ and $|Y_0|$.

The deformation of the surface of the elastic coating under an external action may be characterized by the complex elasticity $K_0 = -\hat{p}/\hat{w}$. Reutov and Rybushkina [7] proposed a membrane model of the response of a single-layer coating, in which the dimensionless elasticity $\bar{K}_0 = K_0 d/(\rho_s c_t^2)$ is represented in the form

$$\bar{K}_0 \approx \bar{m}\alpha^2(\bar{c}_0^2 - \bar{c}^2) - i\gamma_t b_0 \alpha \bar{c}, \tag{3}$$

where $\alpha = kd$ is the dimensionless wavenumber, $\bar{c} = c/c_t$ is the dimensionless phase velocity, $\bar{m}$, $\bar{c}_0$, and b_0 are functions of α, which have the meaning of the parameters of the effective membrane, and γ_t is the parameter of losses in the coating (the coefficient b_0 is independent of γ_t and is related to the coefficient $\bar{b}$ introduced in [7] by the relation $b_0 = \bar{b}/\gamma_t$).

Determination of the nonlinear response of the elastic layer is a labor-consuming problem in which it is necessary to take into account the nonlinearity of the strain tensor and the second harmonic of elastic fields. Since the nonlinearity of the strain tensor is characterized by the parameter $ka \ll 1$, we may assume that it plays a secondary role as compared to the "large" (in the above-mentioned meaning) nonlinearity of the TBL response. Therefore, we confine ourselves to evaluating the deformational nonlinearity of the coating, which confirms these considerations.

We replace the coating by a plate attached to a distributed spring base [3]. Using the thin-plate approximation, we can write the Kármán equation for a periodic flexure of such a model coating (see, e.g., [10]):

$$mw_{tt} - Tw_{xx} + Bw_{xxxx} + K_E w - \left[\frac{6B}{d^2\lambda}\int\limits_x^{x+\lambda}(w_{x'})^2\,dx'\right]w_{xx} + bw_t = -p. \tag{4}$$

Here $m = \rho_s d$ is the surface density of the plate, T is the tension coefficient, $B = Gd^3/[6(1-\mu)]$ is the flexural rigidity ($\mu \approx 0.5$ is Poisson's ratio), K_E is the elasticity of the spring base, b is the absorption factor, and $\lambda = 2\pi/k$ is the flexure period.

Substituting the surface pressure (1) into (4) and defining w as the expansion in powers of $ka \ll 1$, we can easily find the main nonlinear term of this expansion and verify that it does not depend on the presence of the second harmonic in the expression for p if it is of order $O[(ka)^2]$. In this case, the expression for the nonlinear elasticity of the plate (4), with accuracy to terms of order $(ka)^2$, is written in a form similar to (2):

$$\bar{K}_{NL} = -\hat{p}/\hat{w} \approx \bar{K}_0 + \bar{K}_1(ka)^2. \tag{5}$$

Here $\bar{K}_1 = (kd)^2/[2(1-\mu)]$ is the nonlinearity factor. The term $\bar{K}_0$ in expansion (5) is replaced by expression (3).

Using the definitions of Y and $\bar{K}_{NL}$, we can represent the nonlinear dispersion relation for hydroelastic waves in the form

$$\bar{K}_0(\bar{c}, \alpha) + \alpha q V^2 Y_0(\bar{c}, \alpha; V) + [\bar{K}_1(\bar{c}, \alpha) + \alpha q V^2 Y_1(\bar{c}, \alpha; V)](ka)^2 = 0, \tag{6}$$

where $q = \rho_0/\rho_s$ is the ratio of densities of the moving fluid and elastic layer and $V = U/c_t$ is the dimensionless flow velocity. Equation (6) is the generalization of the dispersion relation of the linear theory [7] to the case of weakly nonlinear hydroelastic waves.

2. Calculation of Nonlinear Complex Elasticity of the Boundary Layer. To calculate the nonlinear elasticity of the flow Y we use the results of [8]. Within the framework of approximations described in Sec. 1, Reutov [8] obtained a system of Prandtl equations for the stream function and vorticity of the mean (over the waviness period) TBL flow, which contains wave stresses of second order in the small slope of the surface $ka \ll 1$. The quasi-linear system of equations for complex profiles of the first harmonics of the stream

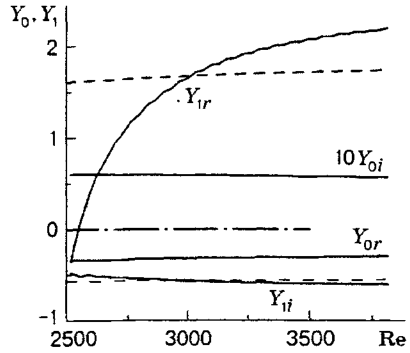

Fig. 2

function and vorticity coincides with that derived in [9] for a parallel flow in the atmospheric TBL. To solve the Prandtl equations, Reutov [8] proposed to use a two-scale approach (approximation of "local similarity"), which leads to a boundary-value problem in ordinary derivatives, which determines the local structure of the TBL in an arbitrary cross section along x. The mean flow in the TBL is characterized by the dynamic velocity u_*, displacement thickness δ^*, local Reynolds number $\mathrm{Re} = U\delta^*/\nu_0$ (ν_0 is the kinematic molecular viscosity of the fluid), and dimensionless pressure gradient $\beta_1 = (\delta^*/(\rho_0 U^2))dP/dx$ (P is the external-flow pressure).

The problem of hydroelastic instability is usually posed for small-scale waviness ($k\delta \gg 1$). The calculations of [8] show that the action of such waviness on the mean flow is mainly localized in the near-wall region ($\eta \ll \delta$). Therefore, the self-similar mean flow in the external region of the TBL (wake region) remains self-similar with origination of waviness. The mean flow in the near-wall region is locally parallel and is described by the equations for a parallel flow obtained by Reutov and Troitskaya [9]. This agrees with the classical notions that the presence of roughness changes the constant in the logarithmic law of the wall, which finally leads to a change in the drag coefficient.

The numerical procedure for solving the boundary-value problem proposed in [8] allows one, from the known "input" parameters Re, β_1, $k\delta^*$, and c/U, to determine the "output" parameters u_*/U, $d\delta^*/dx$, the drag coefficient, and Y. In accordance with the TBL theory on a flat surface [11], a self-similar wake flow exists only if there are negative pressure gradients whose absolute values are not too large: $\beta_1 \geqslant -0.5(u_*/U)^2$.

The approach implemented in [8] allows the calculation of the nonlinear response of the TBL for a fixed value of Re. However, when the waviness is introduced, the values of δ^* and Re do not remain constant for $x = \text{const}$ and depend on the history of TBL expansion (Fig. 1). The problem of the effect of the increment δ^* on the value of Y is also important because the amplitude of waviness under actual conditions may vary along x.

To estimate the effect of the increment δ^*, we performed a selective direct solution of the complete system of Prandtl equations in partial derivatives. The transition to normalized variables was performed in the same way as in [8], but the derivatives with respect to x were retained in the equations. The solution was found by the method of lines [11]. The scheme of discretization along x was borrowed from [12]. The boundary-value problem in ordinary derivatives arising at each step along x was solved by an iterative method: however, in contrast to [12], the method of differential sweeping was used, which allows a significant reduction of the step of discretization along x.

Figure 2 shows the behavior of the expansion coefficients of the flow elasticity (2) $Y_0 = Y_{0r} + iY_{0i}$ and $Y_1 = Y_{1r} + iY_{1i}$, which were calculated within the framework of the local approach and using a direct numerical solution of the Prandtl equations [the solid curves refer to the direct solution of the Prandtl equations for the mean flow for $\mathrm{Re}(x_0) = 2500$ and the dashed curves show the calculation within the framework of the local theory; $c/U = 0$ and $k\delta^*(x_0) = 0.67$]. The problem in partial derivatives was solved under the condition that a TBL with a self-similar wake region comes from a flat surface onto the waviness: in the cross section $x = x_0$

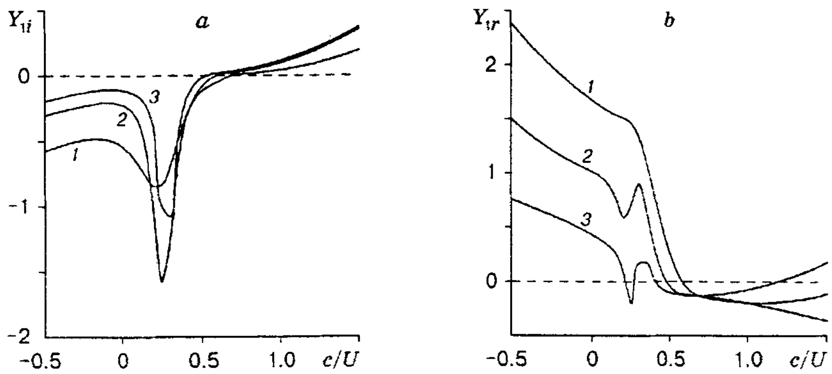

Fig. 3

shown as an example in Fig. 2, we have Re = 2500, $u_*/U = 0.0418$, $\delta/\delta^* = 6.35$, and $d\delta^*/dx = 0.0025$. The local Reynolds number increases downstream almost linearly:

$$\mathrm{Re}\,(x) - \mathrm{Re}\,(x_0) \simeq \mu_1(x - x_0)/\delta^*(x_0). \tag{7}$$

Here $\mu_1 \approx 5.43$. The maximum value of $x - x_0$ for the Re interval shown in Fig. 2 is approximately $40\delta(x_0)$, which corresponds to the length of the elastic insert in the experiments [1, 6]. The TBL thickness increases by a factor of 1.5 over this interval.

As is shown in Fig. 2, the imaginary parts of the nonlinear coefficient Y_1 obtained using the local approach and the direct solution of the Prandtl equations are almost identical. The reason is that Y_{1i} is mainly determined by the velocity profile of the locally parallel mean flow in the buffer region of the TBL. The real part of Y_1 depends to a larger extent on the external flow. Nevertheless, at distances $x - x_0 \approx 15\,\delta(x_0)$, the coefficient Y_{1r} calculated by the direct numerical solution is also close to the value obtained within the framework of the local theory. Actually performed calculations confirm the hypothesis about the weak effect of the increments δ^* on the component of Y_r that is quadratic with respect to ka. Within the framework of the local theory, this result can be explained by the weak dependence of Y_r on the Reynolds number Re. The linear elasticity Y_0 is determined in the same way by direct integration of the Prandtl equations and using the local theory. The data in Fig. 2 refer to slow (divergent) waves with typical phase velocities $c \leqslant 0.05\,U$ [6]. Similar results were obtained for fast waves with phase velocities $c \approx (0.3\text{–}0.4)U$ [1].

Thus, the second-order terms of expansion of the flow elasticity over the slope of the surface, which were obtained within the framework of the local approach, differ insignificantly from the actual values; therefore, the main calculations of Y_0 and Y_1 were performed using the local approach. It was found that Y_0 and Y_1 depend weakly on the introduction of the negative pressure gradient $\beta_1 < 0$ typical of experiments in hydrochannels. Such a dependence becomes significant only for β_1 close to the limiting value given above. The results presented below refer to the case $\beta_1 = 0$.

For the linear part of the elasticity Y_0, Reutov and Rybushkina [7] proposed a quasi-potential approximation of the form

$$Y_0 \simeq -\Big(\frac{c}{U} - f\Big)^2 + \delta Y_0, \tag{8}$$

where $f < 1$ is the parameter of reduction of static elasticity of the potential flow. Analytical approximations of the dependence f on $k\delta^*$ and Re and the calculation results for the dissipative component of elasticity δY_0 are also presented in [7].

Figure 3a and b shows the calculated data for the dependence of the imaginary and real parts of the nonlinear parameter Y_1 on the dimensionless phase velocity of disturbances for Re = 3000 (curves 1–3 correspond to $k\delta^* = 1$, 3, and 6). It is seen that nonlinearity increases the real part of flow elasticity ($Y_{1r} > 0$ for rather small c) and decreases the positive imaginary part of elasticity determining the energy income from the TBL to the wavy surface ($Y_{1i} < 0$ for $Y_{0i} > 0$).

3. Nonlinear Stabilization of Instability. We study the generation of quasi-monochromatic, finite-amplitude waves on the basis of the Landau evolution equation, which allows one to describe the nonlinear stage of development of weak instabilities.

Using relations (3) and (8), we present the nonlinear dispersion equation (6) in the following form:

$$D(\Omega,\alpha;V) - \alpha^2[\bar{K}_1(\Omega,\alpha) + \alpha q V^2 Y_1(\Omega,\alpha;V)]A^2 = 0,$$
$$D = \bar{m}(\Omega^2 - \alpha^2\bar{c}_0^2) + \frac{q}{\alpha}(\Omega - fV\alpha)^2 + i\gamma_t b_0\Omega - \alpha q V^2 \delta Y_0. \qquad (9)$$

Here $\Omega = \alpha\bar{c}$ is the dimensionless frequency and $A = a/d$ is the dimensionless amplitude of the wave. As in the linear theory [7], Eq. (9) contains implicitly two dimensionless parameters, $\mathrm{Re}_t = c_t\delta^*/\nu_0$ and d/δ^*.

Theory of Small Supercriticality. Hydroelastic instability appears when the flow velocity V is greater than the critical value V_c. Waves with frequency Ω_s and wavenumber α_c are excited in the flow–coating system. The calculations of [7] show that the dispersion equation of the linear problem, which is obtained from (9) for $A = 0$ describes the transition from the TWF to wave divergence with increasing losses in the coating γ_t. We obtain the Landau equation within the framework of the asymptotic theory, which is constructed in terms of the small parameter $\varepsilon = (V - V_c)/V_c \ll 1$.

We introduce the dimensionless coordinate $x_1 = x/d$ and time $t_1 = c_t t/d$, multiply the right and left sides of Eq. (9) by $w(x_1, t_1)$, pass from Ω to the spectral operator $\hat{\Omega} = id/dt_1$, and introduce $\delta\hat{\Omega} = \hat{\Omega} - \Omega_c$. We assume that $D(\Omega,\alpha;V) = D_r + iD_i$ in (9) and expand D_r and D_i in the vicinity of the critical point in small $\delta\hat{\Omega} \sim \varepsilon$ and $V - V_c \sim \varepsilon$. Assuming that $A = O(\varepsilon^{1/2})$, we seek the solution of the resultant equation in the form of a series in powers of ε with the main part of the form

$$w(x_1, t_1) = (1/2)\hat{w}(t_1)\exp(i\alpha_c x_1 - i\Omega_c t_1) + \text{c.c.} \qquad (10)$$

From the condition of boundedness of the term of order ε in the expansion of w, we obtain the Landau equation for the complex amplitude $\hat{w}$. Using standard transformations, we pass to the Landau equation for A:

$$\frac{dA}{dt_1} = (\lambda_1 - \lambda_3 A^2)A, \qquad (11)$$

where the coefficients have the form

$$\lambda_1 = \varepsilon\,\frac{D'_{rV}D'_{i\Omega} - D'_{r\Omega}D'_{iV}}{(D'_{r\Omega})^2 + (D'_{i\Omega})^2}\,V_c\Big|_c, \quad \lambda_3 = \alpha^2\,\frac{(\bar{K}_1 + \alpha q V^2 Y_{1r})D'_{i\Omega} - \alpha q V^2 Y_{1i} D'_{r\Omega}}{(D'_{r\Omega})^2 + (D'_{i\Omega})^2}\Big|_c \qquad (12)$$

(the primes denote the derivatives with respect to V and Ω, and the subscript c indicates that the expression is calculated for $\alpha = \alpha_c$ and $\Omega = \Omega_c$).

The nonlinear coefficient λ_3 in (11) was calculated for different γ_t. The parameters of the critical waves from [7, Fig. 8] were used. The function $\lambda_3(\gamma_t)$ has a "plateau" for small γ_t, a small increase for $\gamma_t \sim 0.5$, and tends monotonically to zero as $\gamma_t \to \infty$. The calculations show that the nonlinearity of the coating elasticity $\bar{K}_1$ makes a small contribution to λ_3 as compared to hydrodynamic nonlinearity. An important result of the calculations performed is that λ_3 has a positive value within the entire range of γ_t. Thus, a mild regime of excitation is observed within the limits of applicability of the quasi-linear theory both for fast waves (flutter) and for slow waves (divergence). A steady generation of a traveling wave with amplitude

$$A_* = \sqrt{\lambda_1/\lambda_3} \qquad (13)$$

is established. The theory of small supercriticality is applicable for a small width of the wavenumber-instability band $|\alpha - \alpha_c| \ll \alpha_c$. The amplitudes of steady waves (13) are significantly lower than those actually observed in experiments. To describe more intense waves, we have to reject the expansion in the small supercritical values of ε.

Wave Divergence on a Viscoelastic Coating. Hydrodynamic instability of slow waves is observed on viscoelastic coatings, which are characterized by high losses: $\gamma_t = 1$–100 and $\gamma_t b_0 = 6$–600 [7]. This allows

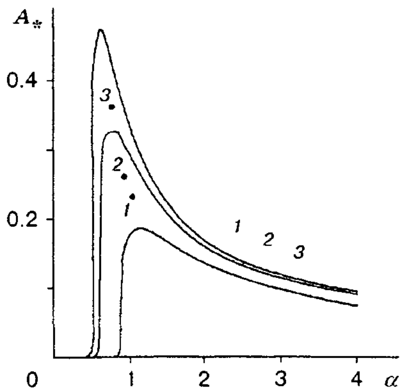

Fig. 4

us to derive an evolution equation using asymptotic expansions in the small parameter $\varepsilon_0 = 1/(\gamma_t b_0) \ll 1$, as was done in [7] for finding the dispersion branches of the linear problem.

We divide the dispersion equation (9) by $\gamma_t b_0$, multiply its right and left sides by $w(x_1, t_1)$, and make the substitution $\Omega \to \hat{\Omega}$. We seek the solution in the form of expansion in the small parameter ε_0 with the main part in the form

$$w(x_1, t_1) = (1/2)\hat{w}(t_1)\exp(i\alpha x_1) + \text{c.c.} \tag{14}$$

In this case, the wave propagation is described by a slow change in the complex amplitude $\hat{w}$, and the expansion of the coefficients of the dispersion equation is performed in terms of $\hat{\Omega} \sim \varepsilon_0$. From the condition of boundedness of the term of order ε_0 in the expansion of w, we obtain the evolution equation for the complex amplitude $\hat{w}(t_1)$, from which follows the Landau equation for A in the form of (11) but with different coefficients:

$$\lambda_1 = \alpha\varepsilon_0(qf^2V^2 - \alpha\bar{m}\bar{c}_0^2)\Big|_{\Omega=0}, \qquad \lambda_3 = \varepsilon_0\alpha^2(\bar{K}_1 + \alpha qV^2Y_{1r})\Big|_{\Omega=0}. \tag{15}$$

It is taken into account in (15) that $\delta Y_{0r} \approx 0\Big|_{\Omega=0}$. The expression for the phase velocity of slow waves has the form

$$\delta\bar{c}_{NL} = \tilde{\lambda}_1 + \tilde{\lambda}_3 A^2, \tag{16}$$

where $\tilde{\lambda}_1 = \varepsilon_0 qV^2\delta Y_{0i}\Big|_{\Omega=0}$ and $\tilde{\lambda}_3 = \varepsilon_0\alpha^2 qV^2Y_{1i}\Big|_{\Omega=0}$. We note that the proposed system for deriving the evolution equation is applicable for an arbitrary nonlinearity of the TBL response, since in this case the nonlinearity should be small only because of the use of expansion (2).

From the expression for the coefficient λ_1, it follows that the origin of instability at high losses in the coating is related to the fact that the absolute value of the real part of flow elasticity is greater than the static elasticity of the coating. The nonlinear increment of flow elasticity and the nonlinear allowance for coating elasticity exert a stabilizing effect on this instability since $Y_{1r} > 0$ (Fig. 3b) and $\bar{K}_1 > 0$. This instability is independent of the imaginary part of flow elasticity Y_i determining the energy flux from the TBL to the wave [8]. The energy interpretation of the instability mechanism cannot be applied here, since the main contribution to the dynamic response of the coating at high losses is made by the dissipative component of its elasticity [the term $-i\gamma_t b_0\alpha\bar{c}$ in (3)]. An increase in losses does not lead to disappearance of instability but only slows down its development. This instability may be characterized as response-resistive [7].

The critical flow velocity is found from the condition $\lambda_1 = 0$ and, in this approximation, it is independent of the losses in the coating. Since $Y_{1r} > 0\Big|_{\Omega=0}$ (Fig. 3b), a steady wave with amplitude (13) is established for $V > V_c$. The calculation results for the amplitude of steady waves versus their wavenumber are plotted in Fig. 4 ($d = 0.32$ cm, $\delta^* = 0.45$ cm, $\text{Re}_t = 350$, and $q = 1$; curves 1–3 correspond to $V = 7$, 9, and 11). The calculations show that the main contribution to A_* is made by the nonlinear part of TBL elasticity (the contribution of $\bar{K}_1$ is small).

The parameters of the TBL and the coating in Fig. 4 correspond to the estimates of [7] obtained for test conditions of [6], where the excitation of divergent waves was studied. The condition of small slopes of the surface was fulfilled in [6] for two or three points at the initial section of experimental dependences of the wave amplitude on the flow velocity. The experimental points 1–3 in Fig. 4 were obtained using the dependences of the wavelength and its amplitude on the flow velocity, which are given in [6, Figs. 12 and 15] for a coating with a shear modulus $G = 5\ \mathrm{N/m^2}$, and correspond to the same values of the dimensionless flow velocity V for which curves 1–3 were constructed.

As is shown in Fig. 4, the wave amplitudes observed in the experiments are reached for moderately small supercritical values where the band of instability (which coincides with the excitation band) becomes wide. A comparison of the theory and experiment allows us to conclude that the experimental points are located near the maxima of the dependences $A_*(\alpha)$. Despite the great width of the excitation band, coherent waves with a clearly expressed period were observed in experiments. The appearance of these waves may be explained by the presence of the processes of nonlinear interaction and by the competition of harmonics growing in the instability band, whose description goes outside the limits of the Landau equation (11). From the analysis of the data in Fig. 4, it follows that a quasi-harmonic wave whose amplitude is close to maximum in the instability band "survives" as a result of competition and interaction of harmonics of the wave packet.

For the family of curves 1–3 in Fig. 4, the critical parameters are $V_c = 5.25$ and $\alpha_c = 2.4$ ($A_* \to 0$ as $V \to V_c$). As the flow velocity V increases, the maximum of the increment is shifted toward increasing α ($\alpha_{\max} = 2.6$, 2.7, and 2.8 for $V = 7$, 9, and 11, respectively). At the same time, in the experiments of [6], the critical wavenumber was actually determined as the wavenumber of the observed wave with the least amplitude (point 1 in Fig. 4). It follows from the data presented that the value of this critical wavenumber is approximately two times smaller than α_c obtained in the linear problem. This circumstance was also noted in [7] and could not be explained within the framework of the linear theory. Thus, a comparison of the theory and experiment allows one to solve the problem of "sampling" the disturbances relative to their wavenumber.

Nonlinear Flutter on a Coating with Small Losses. In the case of small losses in the coating, the effects of dissipation and nonlinearity may be taken into account as small perturbations. The corresponding small parameter $\varepsilon_1 \ll 1$ for the dispersion equation (9) may be introduced in the form of the ratio of the greatest quantity among $\gamma_t b_0 \alpha \bar{c}_0$, $\alpha q V^2 |\delta Y_0|$, $\alpha^2 |\bar{K}_1| A^2$, and $\alpha^3 q V^2 |Y_1| A^2$ to the value $\bar{m}\alpha^2 \bar{c}_0^2$ characteristic of D. The necessary condition for the existence of such a small parameter is the smallness of losses in the coating ($\gamma_t b_0 \ll 1$). Assuming that the small parameter ε_1 exists, we will not identify it explicitly in (9) for brevity.

We denote the function D for $\delta Y_0 = 0$ and $\gamma_t = 0$ as D_0. The dispersion equation $D_0(\Omega, \alpha) = 0$ determines two families of waves of the linear conservative problem:

$$\Omega_{1,2} = \frac{\alpha}{\bar{m}\alpha + q}\left[qVf \pm \sqrt{(\alpha\bar{m} + q)\alpha\bar{m}\bar{c}_0^2 - \alpha q \bar{m} f^2 V^2}\,\right]. \tag{17}$$

As is shown in [7], for a TBL over a coating with small losses, the least critical velocity of the flow is observed for the instability of fast waves lying on the branch $\Omega_1(\alpha)$.

To obtain the evolution equation for the TWF, we introduce $\delta\hat{\Omega} = \hat{\Omega} - \Omega_0$, where $\Omega_0 = \Omega_1(\alpha)$. In this case, in (9) we have an expansion in terms of $\delta\hat{\Omega} \sim \varepsilon_1$ in the vicinity of $\hat{\Omega} = \Omega_0$. The solution for w is sought in the form of an expansion in powers of ε_1 with the main part of the form

$$w(x_1, t_1) = (1/2)\hat{w}(t_1)\exp(i\alpha x_1 - i\Omega_0 t_1) + \text{c.c.} \tag{18}$$

As a result, we obtain the Landau equation for A in the form of (11) with the coefficients

$$\lambda_1 = (\alpha q V^2 \delta Y_{0i} - \gamma_t b_0 \Omega_0)/D'_{0\Omega}\Big|_{\Omega=\Omega_0}, \qquad \lambda_3 = -\alpha^3 q V^2 Y_{1i}/D'_{0\Omega}\Big|_{\Omega=\Omega_0}, \tag{19}$$

where $D'_{0\Omega} = 2\Omega_0(\bar{m} + q/\alpha) - 2qfV$ is the derivative of D_0 with respect to Ω. The allowance for the phase velocity of conservative waves has the form of (16), where $\tilde{\lambda}_1 = qV^2\delta Y_r/D'_{0\Omega}\Big|_{\Omega=\Omega_0}$ and $\tilde{\lambda}_3 = \alpha(\bar{K}_1 + \alpha q V^2 Y_{1r})/D'_{0\Omega}\Big|_{\Omega=\Omega_0}$.

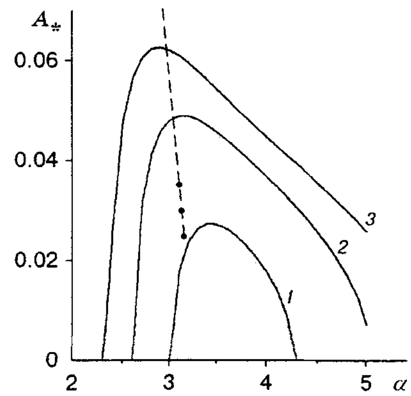

Fig. 5

We can show that the sign of wave energy in the conservative problem (17) coincides with the sign of $\Omega_0 D'_{0\Omega}$. Since we have $D'_{0\Omega} > 0$ and $\Omega_0 > 0$ for fast waves, their energy is positive. It follows from Eq. (19) for λ_1 that destabilization of fast waves is possible for $Y_{0i} > 0$, i.e., in the presence of an energy flux to the wave from the mean TBL flow (Miles mechanism). As is shown in [7], the sign of Y_{0i} becomes positive as the flow velocity increases. Since nonlinearity exerts a stabilizing effect on this instability ($\lambda_3 > 0$ for $Y_{1i} < 0$; see Fig. 3a), there arises a steady wave with amplitude (13). Note that the critical flow velocity and the parameters of fast waves at the threshold of instability, which were found within the framework of the approximate equation (11) with coefficients (19), are close to those obtained numerically in [7] in the case of small losses.

The calculation results of the amplitude of steady waves versus their wavenumber are shown in Fig. 5 for $d = 0.32$ cm, $\delta^* = 0.41$ cm, $\mathrm{Re}_t = 1113$, $\gamma_t = 0.014$, and $q = 1$ (curves 1–3 correspond to $V = 2.8$, 2.9, and 3.0). Origination of the TWF on a coating in the TBL was studied experimentally in [1]. In this case, in comparison of the theory and experiment, because of a stronger scatter of experimental points for the TWF, we used smoothed dependences of the amplitude and wavelength of the flow velocity, which were given in [1, Figs.13 and 14] for a coating with $G = 74$ N/m^2. Taking these dependences into account, we constructed an averaged curve of experimental data (dashed curve in Fig. 5). Points corresponding to $V = 2.8$, 2.9, and 3.0 were placed on this curve (Fig. 5). The parameter of losses γ_t was chosen so that the theoretical curve was closer to the experimental point corresponding to the least value of V.

The curve of experimental data in Fig. 5 is located near the maxima of the theoretical dependences $A_*(\alpha)$. In this case, we have $V_c = 2.77$ and $\alpha_c = 3.6$. The wavenumber in the maximum of the increment of linear instability increases with increasing V ($\alpha_{\max} = 3.6$, 3.8, and 4.0 for $V = 2.8$, 2.9, and 3.0, respectively). As follows from these data, the shift in terms of the wavenumber between the maximum of the increment and the experimental point with the least value of A_* is insignificant in this case. Therefore, the critical wavenumber for the TWF measured in the experiments agrees with the calculation within the framework of the linear theory [7]. Comparing the theoretical and experimental data for the TWF, we may assume that, as in the case of slow waves, the nonlinear processes of competition and interaction of the harmonics of the packet lead to the "survival" of the harmonic whose amplitude is close to maximum in the instability band. As is shown in Fig. 5, the calculated amplitude for the TBL increases faster with increasing flow velocity. A possible reason for this disagreement may be the neglect of damping factors (for example, the effect of the second harmonic of the flexure).

Note that the amplitudes of steady waves can be found directly from the nonlinear dispersion equation (9). The numerical solution of Eq. (9) relative to Ω and A yields amplitude dependences that are close to those presented in Figs. 4 and 5. The transition to the Landau equation (11) used above allowed us to seek the solution of the nonlinear problem taking into account the known results of the linear theory and reveal the mechanisms of stabilization of hydroelastic instability. In addition, the proposed scheme of derivation of the Landau equation for coatings with small and large losses can be further extended to the case of a multiwave flexure of the surface (excitation of wave packets).

Conclusions. A nonlinear theory of generation of weak quasi-monochromatic waves on the surface of an elastic coating in a turbulent boundary layer of an incompressible fluid flow is developed in this paper. The nonlinear dispersion equation is formulated in terms of the complex elasticity of the flow and the coating, which allows the maximum use of the results of solving the linear problem considered in the previous work of the authors [7]. It is shown that, for small slopes of the surface ($ka \ll 1$), the limitation of hydroelastic instability is determined by hydrodynamic nonlinearity arising as a result of deformation of the velocity profile of the mean (over the waviness period) flow in the boundary layer under the action of waviness.

A comparison of the resultant weakly nonlinear theory with known experimental data allows us to conclude that the waves with the maximum possible amplitude in the instability band for a given flow velocity "survive." This offers an explanation to the fact that the critical wavenumber of divergent waves in the experiment is significantly (approximately by two times) smaller than its theoretical value obtained within the framework of the linear theory, whereas this difference for the TWF is small.

At the same time, it remains unclear why waves with certain finite amplitudes were observed in experiments after the loss of stability (in fact, there is no region of very small supercriticality in the graphs). We also note that, in the case of wave divergence, the approximation of the quasi-monochromatic wave becomes rapidly invalid as the flow velocity increases, since multiple resonant harmonics of the primary disturbance fall within the instability band.

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